Hyperbolic Orbit Equations with Asymptote-Based Coordinates

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Theme

BASIC relations between position, velocity, and time in hyperbolic orbits are derived for polar and cartesian coordinates referenced to one of the hyperbola asymptotes. The polar coordinate system angle is measured from the asymptote direction, and the cartesian coordinates are along and normal to the asymptote in the orbit plane. The resulting relations are simple in form, allowing easy formula manipulation and direct computation of all state variables.

These coordinate systems may be advantageous for applications such as the estimation of planet encounter and escape trajectories, where one asymptote direction often is better defined than periapsis location. Polar coordinates probably are most convenient for processing onboard planet sightings, while cartesian variables are more suitable with ground tracking data. The orbit equations permit both coordinate frames, and hence both types of navigation data, to be used together.

Contents

The orbit geometry is shown in Fig. 1. The primary coordinate systems, with origins at the mass center, are r, θ—polar coordinates with radius r, and angle θ measured from the asymptote direction; x, y—Cartesian variables with x along and y perpendicular to the asymptote. The variables $\tilde{r}$ and $\tilde{x}$ in Fig. 1. defined by extending the radius to the asymptote, are closely related to several of the state variables. The orbit equations use three other variables—R, X and l—which will be defined shortly.

The orbit parameters used in the equations are V_∞ = asymptotic velocity; $\bar{a} = -a = \mu/V_\infty{}^2$ = negative of the semimajor axis (μ = gravitational constant of the central body); b = asymptote offset from the mass center = semiminor axis.

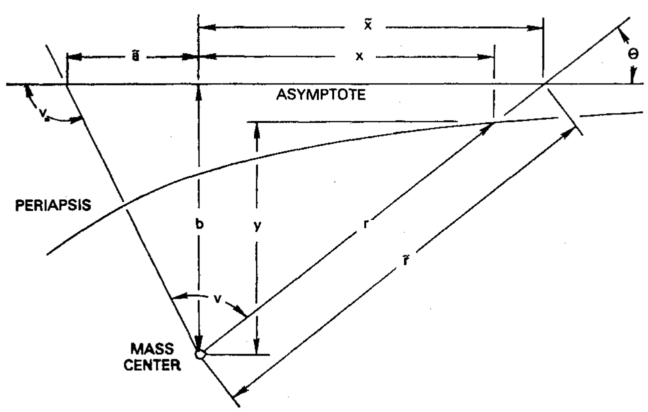

Fig. 1 Orbit geometry.

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The quantities defined previously are related to the usual parameters p (semilatus rectum), e (eccentricity) and variables v (true anomaly) and H (hyperbolic anomaly) by

$$p = b^2/\bar{a} = \bar{a}(e^2 - 1),\ v = v_\infty - \theta,\ (\bar{a}^2 + b^2)^{1/2}\cosh(H) = r + \bar{a}$$

where v_∞ is the asymptotic value of v, defined by $e \cos v_\infty = -1$, $e \sin v_\infty = b/\bar{a}$. The parameters $\bar{a}, b$ and angles v, v_∞ are also shown in Fig. 1.

The orbit equations for r,θ and x,y coordinates are derived from two general velocity formulas, which are given below to allow the basic relations to be derived from the material in the synoptic.

Let u be the component of r in the direction at a fixed angle ϕ from periapsis, and let $\delta = \phi - v$, so that $u = r \cos\delta$. Denote velocity by a dot over a variable. Differentiating u and making appropriate substitutions yields

$$r\dot{u}/V_\infty = \bar{a}eu \sin v/b + b \sin\delta \tag{1}$$

$$= \pm(u^2 - 2\bar{a}e \cos\phi\, u + b^2)^{1/2} \tag{2}$$

Substituting the proper values of ϕ into Eq. (2) for the r,θ and x,y velocity components gives

$$\dot{r} = \pm V_\infty(r^2 + 2\bar{a}r - b^2)^{1/2}/r, \quad r\dot{v} = -r\dot{\theta} = \pm bV_\infty/r \tag{3}$$

$$\dot{x} = \pm V_\infty(x^2 + 2\bar{a}x + b^2)^{1/2}/r, \quad \dot{y} = \pm V_\infty(b - y)/r \tag{4}$$

The quadratic forms in $\dot{r}$ and $\dot{x}$ appear frequently in the orbit relations. For compactness these are denoted by

$$R = (r^2 + 2\bar{a}r - b^2)^{1/2}, \quad X = (x^2 + 2\bar{a}x + b^2)^{1/2} \tag{5}$$

so that

$$\dot{r} = \pm V_\infty R/r, \quad \dot{x} = \pm V_\infty X/r.$$

The key identity on which the succeeding orbit relations depend is derived from Eqs. (1–5). This identity also introduces the third new variable l

$$l = r + R = x + X = b \cot(\theta/2) = \tilde{r} + \tilde{x} = (\bar{a}^2 + b^2)^{1/2}\exp(H) = -\bar{a} \tag{6}$$

Equation (6) is valid for the entire orbit when the variables are given correct signs. When $\theta > v_\infty$, $R < 0$; and if $\bar{a} > b$, $X < 0$ when $\sin \theta < -b/\bar{a}$. All velocity components in Eqs. (3) and (4) then take the same sign, determined by whether the coordinate base is the departure (+) or approach (−) asymptote. Also, when $\theta > 180°$, $\tilde{r} < 0$ and $\tilde{x} = \tilde{r} \cos\theta$.

Using Eqs. (3–6), all state variables can be expressed as functions of l, $\bar{a}$, b, and V_∞. These formulas are a convenient way of describing the orbit variables and allow relations between them to be easily found. The position coordinates and R (which appears in the time formula) are

$$\begin{aligned} r &= (l^2 + b^2)/2(l + \bar{a}), \quad R = (l^2 + 2l\bar{a} - b^2)/2(l + \bar{a}) \\ \sin\theta &= 2bl/(l^2 + b^2), \quad \cos\theta = (l^2 - b^2)/(l^2 + b^2) \\ x &= (l^2 - b^2)/2(l + \bar{a}), \quad y = bl/l + \bar{a}) \\ \tilde{r} &= (l^2 + b^2)/2l, \quad \tilde{x} = (l^2 - b^2)/2l \end{aligned} \tag{7}$$

The velocity components of interest are

$$\dot{r} = \pm V_\infty(l^2 + 2l\bar{a} - b^2)/(l^2 + b^2), \quad r\dot{\theta} = \mp V_\infty 2b(l + \bar{a})/(l^2 + b^2)$$

$$\dot{x} = \pm V_\infty(l^2 + 2l\bar{a} + b^2)/(l^2 + b^2), \quad \dot{y} = \pm V_\infty(2\bar{a}b)/(l^2 + b^2) \tag{8}$$

$$\dot{l} = \pm V_\infty 2(l + \bar{a})^2/(l^2 + b^2)$$

The orbit relations are completed by rearranging and integrating the $\dot{r}$ formula of Eq. (3) to obtain the time since pericenter

$$t - t_p = \pm 1/V_\infty[R - \bar{a}\log(r + R + \bar{a}) + (\bar{a}/2)\log(\bar{a}^2 + b^2] = \pm 1/V_\infty[(l^2 + 2l\bar{a} - b^2)/2(l + \bar{a}) - \bar{a}\log(l + \bar{a}) + (\bar{a}/2)\log(\bar{a}^2 + b^2)] \tag{9}$$

Many explicit relationships between variables can be derived from Eqs. (6–8), for example

$$\tilde{r}/r = \tilde{x}/x = b/y = (l + \bar{a})/l$$

$$y = b - \bar{a}b/(x + X + \bar{a})$$

$$r - x = X - R = b^2/(l + \bar{a}); \quad R - x = X - r = \bar{a}y/b$$

$$\pm\dot{r}/V_\infty = R/r = (\tilde{x} + \bar{a})/\tilde{r}; \ \pm\dot{x}/V_\infty = X/r = (\tilde{r} + \bar{a})/\tilde{r}$$

Time may be expressed in x,y coordinates as

$$t - t_p = \pm 1/V_\infty[x + \bar{a}y/b - \bar{a}\log(x + X + \bar{a}) + (\bar{a}/2)\log(\bar{a}^2 + b^2)]$$

More generally, every orbit variable can be explicitly written in terms of any of the variable pairs $(r,R)\ldots(\tilde{r},\tilde{x})$ by substituting the equivalent expression of Eq. (6) for l in Eqs. (7) and (8). Computation of the complete orbit state vector from any variable in Eq. (6) requires only one square root operation, to determine the complementary variable defining l.

The velocity formulas of Eqs. (1) and (2) are valid for all orbits by substituting $V_\infty\bar{a}/b = (\mu/p)^{1/2}$. Analogues of Eq. (6) can be obtained for elliptic orbits, but do not provide simple expressions for the state variables such as Eqs. (7) and (8). Such analogues therefore have little apparent utility.

Approximate relations between r,θ and time for large and moderately large eccentricity orbits are obtained by noting that $\dot{r}$ is nearly constant at radii $\geq p/2$; $\dot{r} = \pm V_\infty$ at $r = p/2$, and reaches its maximum absolute value, at $r = p$, of $V_\infty e/(e^2 - 1)^{1/2}$, which is only slightly greater than V_∞ for reasonably high eccentricities. Setting $\dot{r} \approx \pm V_\infty$ (which implies $R \approx r$) yields the further approximations:

$$\tilde{r} \approx r + \bar{a}/2, \quad \sin\theta = b/\tilde{r} \approx b/(r + \bar{a}/2)$$

Simplified analyses of onboard approach navigation based on these approximate orbit relations provide error estimates within a few percent of those obtained using the exact orbit equations. Realistic cutoff points for approach navigation are within the region of validity of the approximations. The $\tilde{r}$ approximation also suggests a simple method of drawing the portions of a hyperbola with radii $\geq/p2$.